Math 249 Lecture 37 Notes

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November 20, 2017

1 Chevalley's Theorem

1.1 Syzygies of a module

If we have a module over a ring, it'll satisfy some relations.

Definition 1.1. A *syzygy* of an R-module M with generators $x_1, \dots, x_n$ is a relation $a_1 x_1 + \cdots + a_n x_n = 0$.

The following lemma will help us prove Chevalley's theorem, but it is just a general fact about rings. You should think of it as saying something about syzygies.

Lemma 1.1. *If $h_1, \dots, h_m \in R^G$ are homogeneous, h_1 is not in the ideal of R^G generated by $h_2, \dots, h_m$, and $g_1 h_1 + \cdots + g_m h_m = 0$, where $g_i \in R$ are homogeneous, then $g_1 \in I_G$.*

Proof. Notice that h_1 is not in the ideal of R generated by $h_2, \dots, h_m$. Why? Suppose $h_1 = a_2 h_2 + \cdots + a_m h_m$. Apply the Reynolds operator $R = \frac{1}{|G|} \sum_g g$ to get

$$h_1 = k(a_2) h_2 + \cdots + k(a_m) h_m,$$

where $k \in R^G$. Now induct on the degree of g_1. If $\deg g_1 = 0$, then $g_1 = 0$. If $\deg g_1 = d > 0$, then apply ∂_i to $g_1 h_1 = \cdots + g_m h_m = 0$ to get a new relation of lower degree, $\partial_i(g_1) h_1 + \cdots$. By induction, $\partial_i(g_1) \in I_G$ for all i. Then $\partial_i g_1 = 0$ in R/I_G for all i, so $g_1 = 0$ in R/I_G. Then $g_1 \in I_G$. $\square$

1.2 Proof of Chevalley's theorem

We will prove these parts of Chevalley's theorem:

Theorem 1.1 (Chevalley)**.** *If $G \circlearrowright \mathbb{R}^n$ is a Coxeter group, then*

1. *R^G is a polynomial ring $\mathbb{R}[f_1, \dots, f_r]$; i.e. $f_1, \dots, f_r$, the homogeneous minimal generators of I_G are algebraically independent.*

2. $R = \mathbb{R}[x_1, \ldots, x_n]$ *is a free* R^G*-module (of rank* $|G|$*).*

Proof. First let's prove statement 2. We know that R/I_G generates R as an R^G module and that $\dim R/I_G \geq |G|$. We want to show that $\dim R/I_g = |G|$. Let's use divided difference operators. We know that ∂_w preserve I_G and so are well-defined on R/I_G.

On $(R/I_G)_d$, each ∂_w for $\ell(w) = d$ give a map $(R/I_G)_d \to \mathbb{R}$. We claim that these span $(R/I_G)_d^*$. Proceed by induction on d. For $d = 0$, $\partial_1 = \mathrm{id}$, so the base case is satisfied. Given an $f \in (R/I_G)_d$ such that $\partial_w f = 0$ for all $\ell(w) = d$, we want to show that $f = 0$. Look at $\partial_i f$. Let v have $\ell(v) = d - 1$; what is $\partial_v \partial_i f$ equal to?

If $w = vs_i$ with $\ell(w) = d$, then $\partial_v \partial_i f = \partial_w f = 0$. Otherwise, $\ell(vs_i) = d - 2$, so v has a reduced factorization that ends in s_i. In this case, $\partial_v = \partial_u \partial_i$, and we get that $\partial_v \partial_i = \partial_u \partial_i^2 = 0$. So $\partial_v \partial_i f = 0$ for all i, so by induction, $\partial_i f = 0$. Then $f \in (R/I_G)^G$, which are constants, so $f = 0$.

Now, let's prove statement 1 using our lemma from before. We want to show that if $f_1, \ldots, f_n$ is a minimal homogeneous generating set of I_G (or of I_G), then the f_i are algebraically independent. Suppose $h(f_1, \ldots, f_n) = 0$, where $h(y_1, \ldots, y_n)$ is a nonzero polynomial. Without loss of generality, h is homogeneous of degree d relative to $\deg y_i = d_i$.

Let $h_i = \frac{\partial h}{\partial y_i}(f_1, \ldots, f_n)$. By the chain rule, for all k,

$$0 = \frac{\partial}{\partial x_k} h(f_1, \ldots, f_n) = \sum_i h_i \frac{\partial f_i}{\partial x_k}.$$

Without loss of generality, $h_1, h_2, \ldots, h_m$ minimally generate the ideal R^G generated by all the h_i. Then $h_j = \sum_{i=1}^m g_{j,i} h_i$, where $g_{j,i} \in R^G$. Then

$$0 = \sum_i h_i \frac{\partial f_i}{\partial x_k} = \sum_{i=1}^m h_i \left(\frac{\partial f_i}{\partial x_k} + \sum_{j=m+1}^N g_{j,i} \frac{\partial f_j}{\partial x_k} \right).$$

$$\frac{\partial f_1}{\partial x_k} + \sum_{j>m} g_{j,i} \frac{\partial f_j}{\partial x_k} \in I_g = \sum_{i \neq 1} r_i f_i$$

Then $\sum_k x_k \frac{\partial f_1}{\partial x_k} = d_1 f_1 + \sum_{j>m} g - j, i d_j f_j = \sum_{i \neq 1} d_i r - i f_i$, which is a contradiction. □

Corollary 1.1. *Let* $H_{R/I_G}(q)$ *be the generating function for the* $\dim(R/I_G)_d$ *(in variable* q*). Then*

$$H_{R/I_G}(q) = \sum_{w \in G} q^{\ell(w)}.$$

$$\prod \frac{1}{1 - q^{d_i}} \cdot H_{R/I_G}(q) = \frac{1}{1 - q^n}.$$